College of Technological Studies
Department of Civil Engineering Technology

CE 278 Structural Analysis

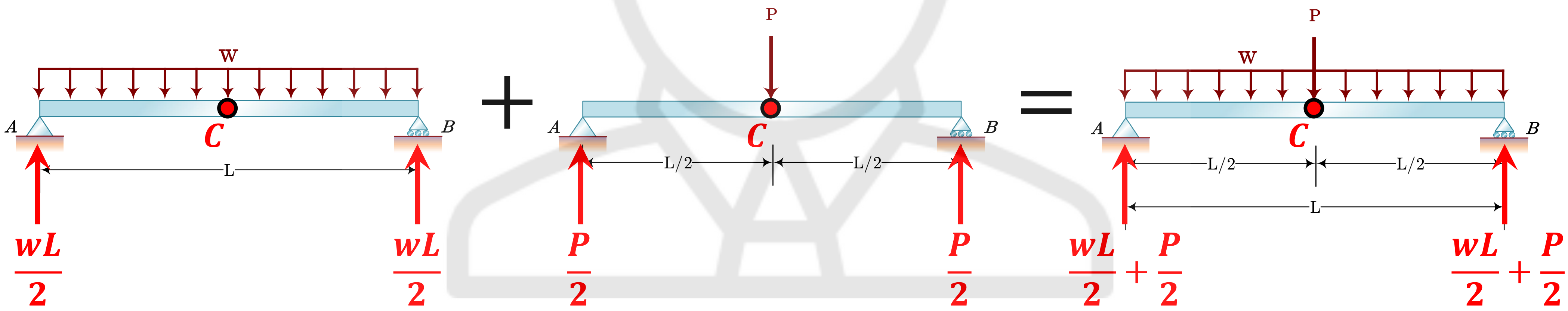
Tutorial (7)

Deflections in Beams

Principle of Superposition:

For a linearly elastic structure, the load effects caused by two or more loadings are the sum of the load effects caused by each loading separately.

Load effects → Stresses – strains – shear forces – bending moments – rotations – deflections



Reactions:

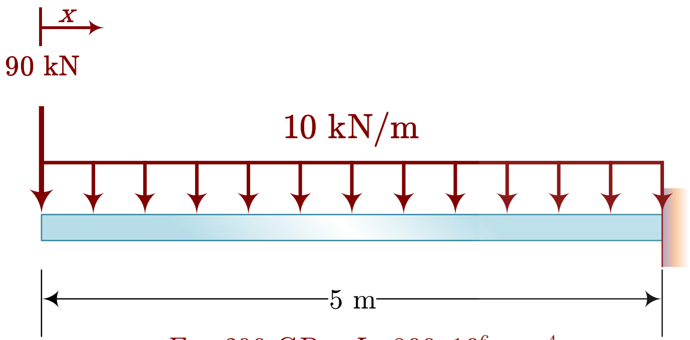
Bending Moment:

$$M_C = \frac{wL^2}{8} \qquad M_C = \frac{PL}{4} \qquad M_C = \frac{wL^2}{8} + \frac{PL}{4}$$

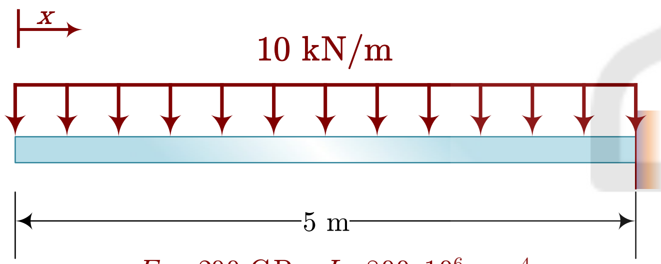
Deflection:

$$\Delta_C = \frac{5wL^4}{384\,EI} \qquad \Delta_C = \frac{PL^3}{48\,EI} \qquad \Delta_C = \frac{5wL^4}{384\,EI} + \frac{PL^3}{48\,EI}$$

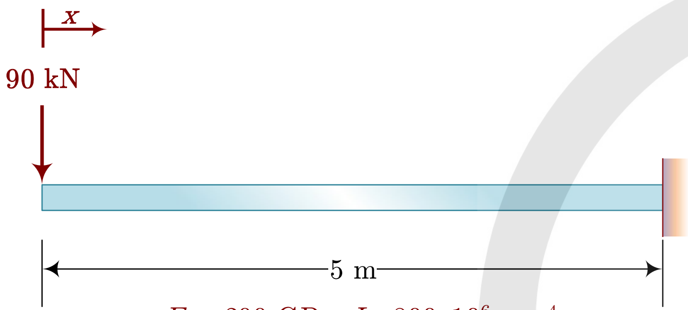
Example (1): Determine the displacement at the free end.



$E = 200$ GPa, $I = 800\text{x}10^6$ mm^4

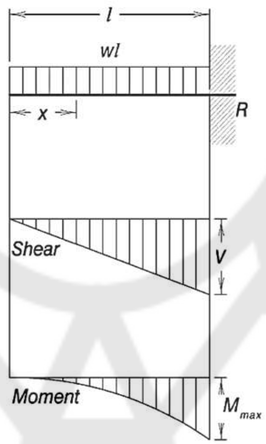


$E = 200$ GPa, $I = 800\text{x}10^6$ mm^4



$E = 200$ GPa, $I = 800\text{x}10^6$ mm^4

19. CANTILEVERED BEAM — UNIFORMLY DISTRIBUTED LOAD



Total Equiv. Uniform Load .... $= 4wl$

$R = V$ .... $= wl$

V_x .... $= wx$

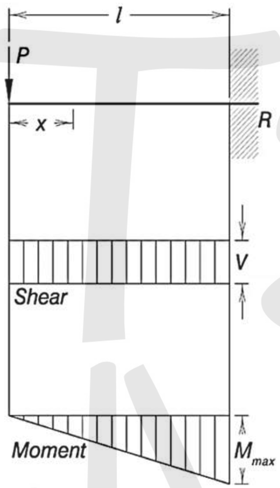
M_{max} (at fixed end) .... $= \frac{wl^2}{2}$

M_x .... $= \frac{wx^2}{2}$

Δ_{max} (at free end) .... $= \frac{wl^4}{8EI}$

Δ_x .... $= \frac{w}{24EI}\left(x^4 - 4l^3x + 3l^4\right)$

22. CANTILEVERED BEAM — CONCENTRATED LOAD AT FREE END



Total Equiv. Uniform Load .... $= 8P$

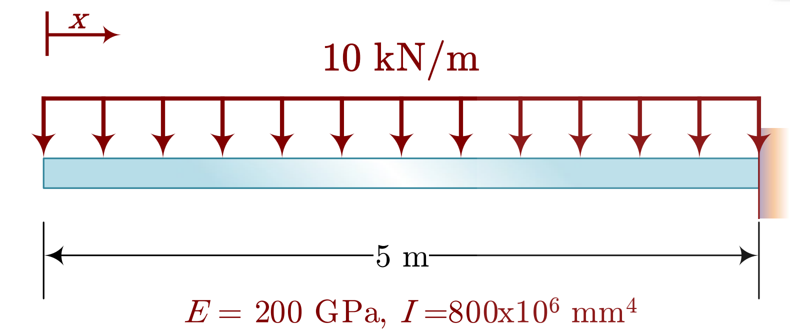
$R = V$ .... $= P$

M_{max} (at fixed end) .... $= Pl$

M_x .... $= Px$

Δ_{max} (at free end) .... $= \frac{Pl^3}{3EI}$

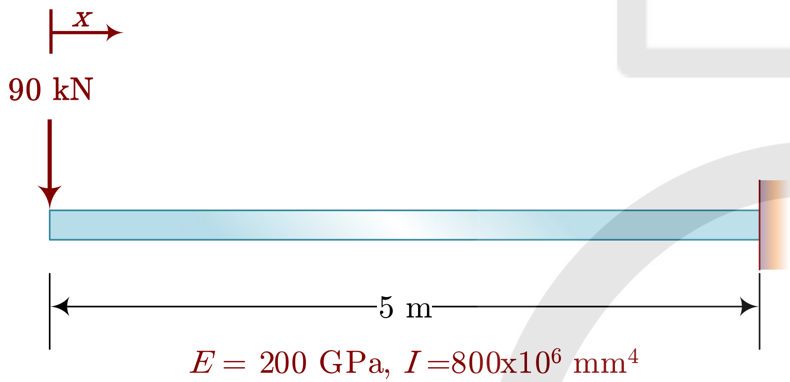
Δ_x .... $= \frac{P}{6EI}\left(2l^3 - 3l^2x + x^3\right)$



$$w = 10\frac{\text{kN}}{\cancel{\text{m}}}\times\frac{1\ \cancel{\text{m}}}{1000\ \text{mm}} = 0.01\frac{\text{kN}}{\text{mm}} \qquad l = 5\ \cancel{\text{m}}\times\frac{1000\ \text{mm}}{1\ \cancel{\text{m}}} = 5000\,\text{mm}$$

$$E = 200\ \text{GPa} = 200\frac{\text{kN}}{\text{mm}^2} \qquad I = 800\times10^6\ \text{mm}^4$$

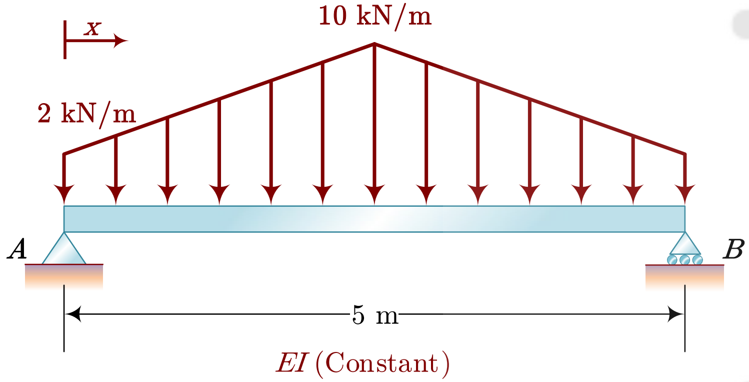
$$\Delta_w = \frac{wl^4}{8EI} = \frac{\left(0.01\frac{\text{kN}}{\text{mm}}\right)(5000\,\text{mm})^4}{8\left(200\frac{\text{kN}}{\text{mm}^2}\right)\left(800\times10^6\ \text{mm}^4\right)} = 4.88\,\text{mm}$$

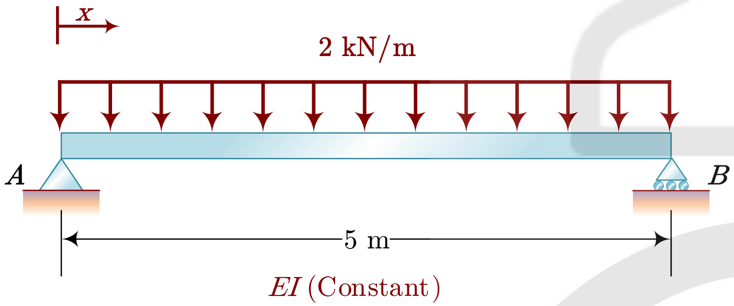


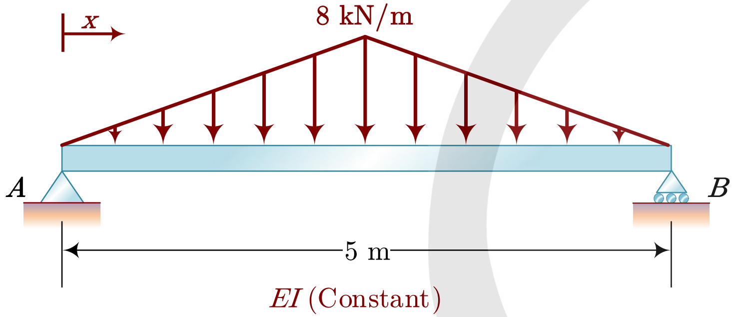
$$\Delta_P = \frac{Pl^3}{3EI} = \frac{(90\,\text{kN})(5000\,\text{mm})^3}{3\left(200\frac{\text{kN}}{\text{mm}^2}\right)\left(800\times10^6\ \text{mm}^4\right)} = 23.43\,\text{mm}$$

$$\Delta = \Delta_w + \Delta_P = 4.88 + 23.43 = \boxed{28.31\,\text{mm}}$$

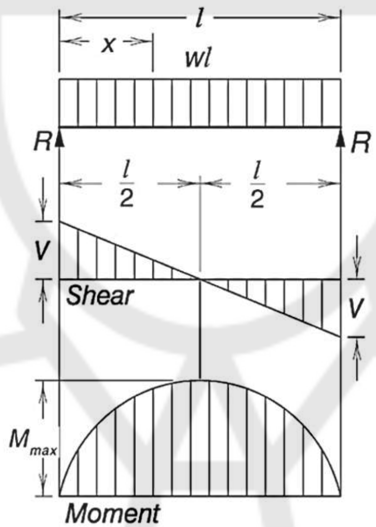
Example (2): Determine the displacement at x = 2 m.





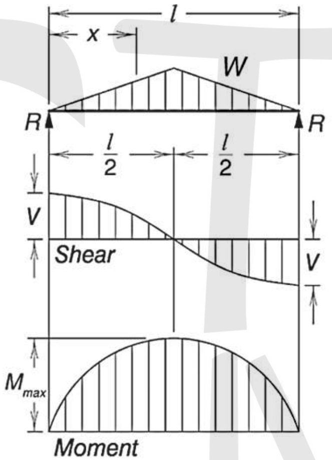


1. SIMPLE BEAM — UNIFORMLY DISTRIBUTED LOAD

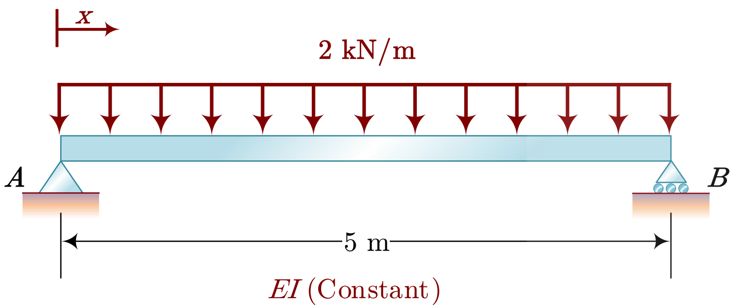


- Total Equiv. Uniform Load .... $= wl$
- $R = V$ .... $= \frac{wl}{2}$
- V_x .... $= w\left(\frac{l}{2} - x\right)$
- M_{max} (at center) .... $= \frac{wl^2}{8}$
- M_x .... $= \frac{wx}{2}(l - x)$
- Δ_{max} (at center) .... $= \frac{5wl^4}{384EI}$
- Δ_x .... $= \frac{wx}{24EI}\left(l^3 - 2lx^2 + x^3\right)$

3. SIMPLE BEAM — LOAD INCREASING UNIFORMLY TO CENTER



- Total Equiv. Uniform Load .... $= \frac{4W}{3}$
- $R = V$ .... $= \frac{W}{2}$
- V_x (when $x < \frac{l}{2}$) .... $= \frac{W}{2l^2}\left(l^2 - 4x^2\right)$
- M_{max} (at center) .... $= \frac{Wl}{6}$
- M_x (when $x < \frac{l}{2}$) .... $= Wx\left(\frac{1}{2} - \frac{2x^2}{3l^2}\right)$
- Δ_{max} (at center) .... $= \frac{Wl^3}{60EI}$
- Δ_x (when $x < \frac{l}{2}$) .... $= \frac{Wx}{480EIl^2}\left(5l^2 - 4x^2\right)^2$



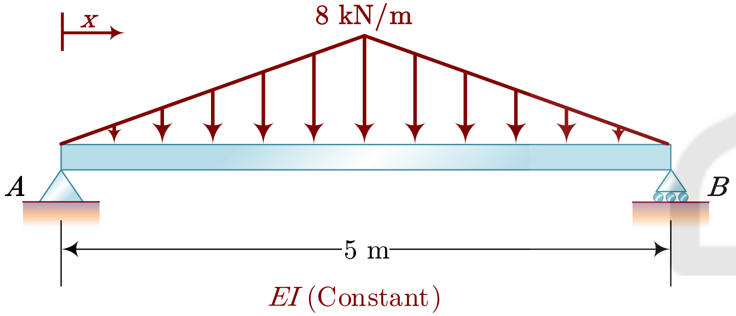
$$w_1 = 2\frac{\text{kN}}{\cancel{\text{m}}} \times \frac{1\ \cancel{\text{m}}}{1000\ \text{mm}} = 0.002\frac{\text{kN}}{\text{mm}} \qquad l = 5\ \cancel{\text{m}} \times \frac{1000\ \text{mm}}{1\ \cancel{\text{m}}} = 5000\,\text{mm}$$

$$x = 2\ \cancel{\text{m}} \times \frac{1000\ \text{mm}}{1\ \cancel{\text{m}}} = 2000\,\text{mm} \qquad EI = \text{Constant}$$

$$\Delta_1 = \frac{w_1 x}{24EI}\left(l^3 - 2lx + x^3\right)$$

$$= \frac{\left(0.002\frac{\text{kN}}{\text{mm}}\right)(2000\,\text{mm})}{24EI}\left((5000\,\text{mm})^3 - 2(5000\,\text{mm})(2000\,\text{mm}) + (2000\,\text{mm})^3\right)$$

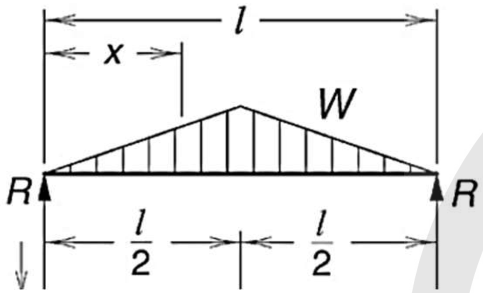
$$= \frac{2.216\times10^{10}}{EI}$$



$$\Delta_2 = \frac{Wx}{480EIl^2}\left(5l^2 - 4x^2\right)^2 = \frac{wlx}{480EIl^2}\left(5l^2 - 4x^2\right)^2$$

$$= \frac{\left(0.008\frac{\text{kN}}{\text{mm}}\right)(5000\,\text{mm})(2000\,\text{mm})}{480EI\,(5000\,\text{mm})^2}\left(5(5000\,\text{mm})^2 - 4(2000\,\text{mm})^2\right)^2$$

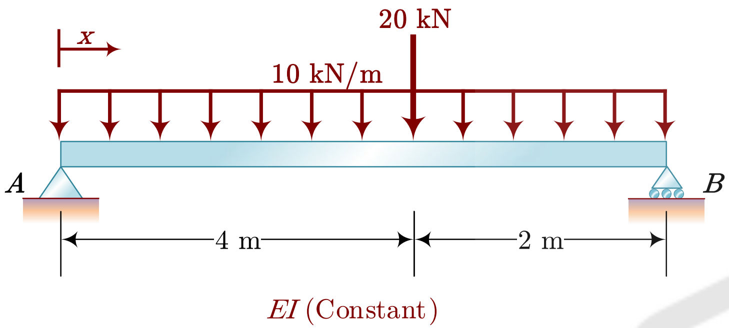
$$= \frac{7.92\times10^{10}}{EI}$$

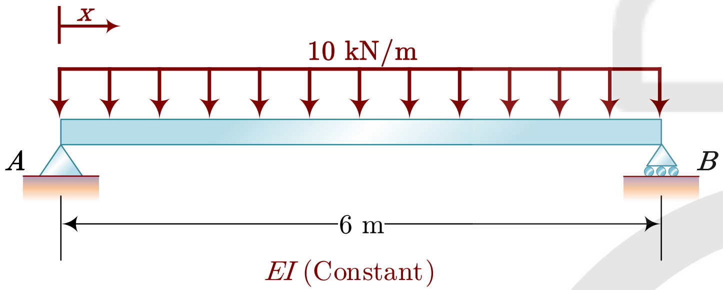


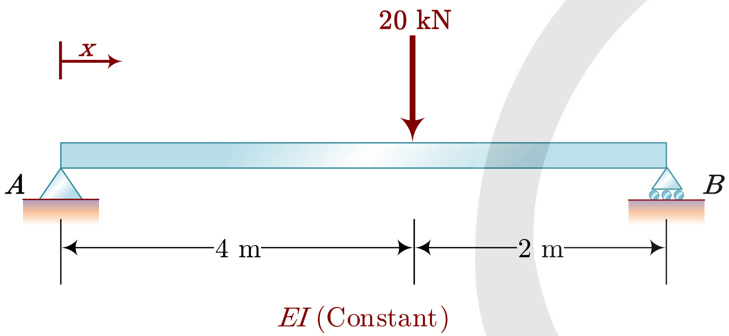
$$W = wl = \left(0.008\frac{\text{kN}}{\cancel{\text{mm}}}\right)(5000\ \cancel{\text{mm}}) = 40\,\text{kN}$$

$$\Delta = \Delta_1 + \Delta_2 = \frac{2.216\times10^{10}}{EI} + \frac{7.92\times10^{10}}{EI} = \boxed{\frac{10.136\times10^{10}}{EI}\,\text{mm}}$$

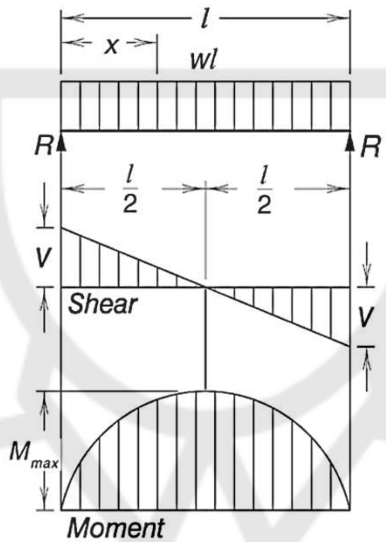
Example (3): Determine the maximum displacement for the beam shown in the figure.







1. SIMPLE BEAM — UNIFORMLY DISTRIBUTED LOAD



Total Equiv. Uniform Load .... $= wl$

$R = V$ .... $= \frac{wl}{2}$

V_x .... $= w\left(\frac{l}{2} - x\right)$

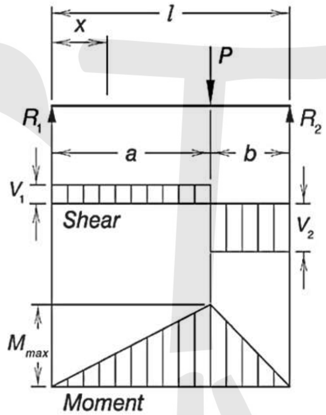
M_{max} (at center) .... $= \frac{wl^2}{8}$

M_x .... $= \frac{wx}{2}(l - x)$

Δ_{max} (at center) .... $= \frac{5wl^4}{384EI}$

Δ_x .... $= \frac{wx}{24EI}\left(l^3 - 2lx^2 + x^3\right)$

8. SIMPLE BEAM — CONCENTRATED LOAD AT ANY POINT



Total Equiv. Uniform Load .... $= \frac{8Pab}{l^2}$

$R_1 = V_1$ ($= V_{max}$ when $a < b$) .... $= \frac{Pb}{l}$

$R_2 = V_2$ ($= V_{max}$ when $a > b$) .... $= \frac{Pa}{l}$

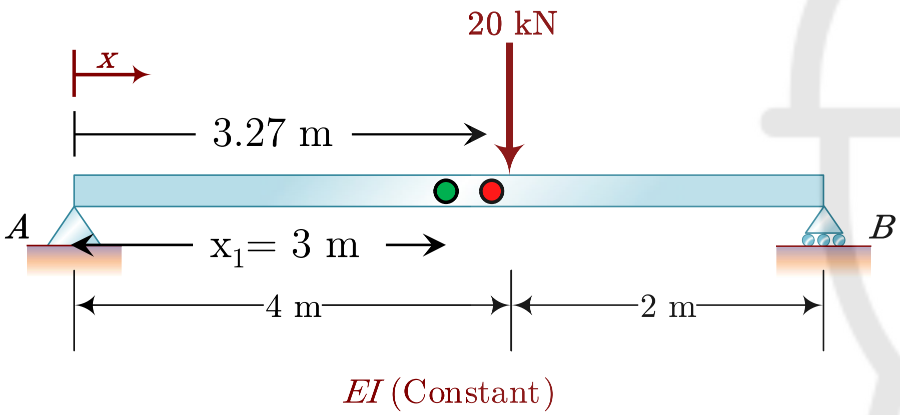
M_{max} (at point of load) .... $= \frac{Pab}{l}$

M_x (when $x < a$) .... $= \frac{Pbx}{l}$

Δ_{max} $\left(\text{at } x = \sqrt{\frac{a(a+2b)}{3}}, \text{ when } a > b\right)$ .... $= \frac{Pab(a+2b)\sqrt{3a(a+2b)}}{27EIl}$

Δ_a (at point of load) .... $= \frac{Pa^2b^2}{3EIl}$

Δ_x (when $x < a$) .... $= \frac{Pbx}{6EIl}\left(l^2 - b^2 - x^2\right)$

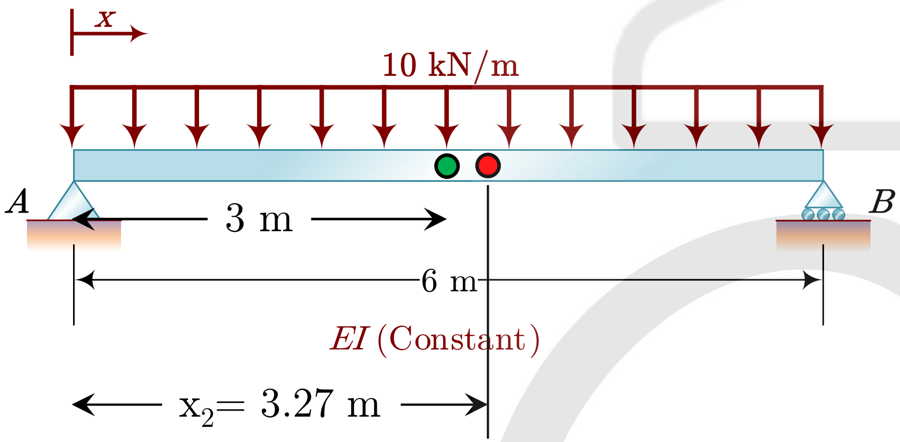


$$x=\sqrt{\frac{a(a+2b)}{3}}=\sqrt{\frac{4(4+2(2))}{3}}=\sqrt{\frac{4(4+2(2))}{3}}=3.27\,\text{m}$$

$$\Delta_{\max}\left(\text{at } x=\sqrt{\frac{a(a+2b)}{3}},\text{ when } a>b\right)=\frac{Pab(a+2b)\sqrt{3a(a+2b)}}{27EIl}$$

$$\Delta_{\max 1}=\frac{7.742\times10^{10}}{EI}$$

$$\Delta_{x1}=\frac{Pbx_1}{6EIl}\left(l^2-b^2-x_1^2\right)=\frac{7.666\times10^{10}}{EI}$$



$$\Delta_{\max 2}=\frac{5wl^4}{384EI}=\frac{1.688\times10^{11}}{EI}$$

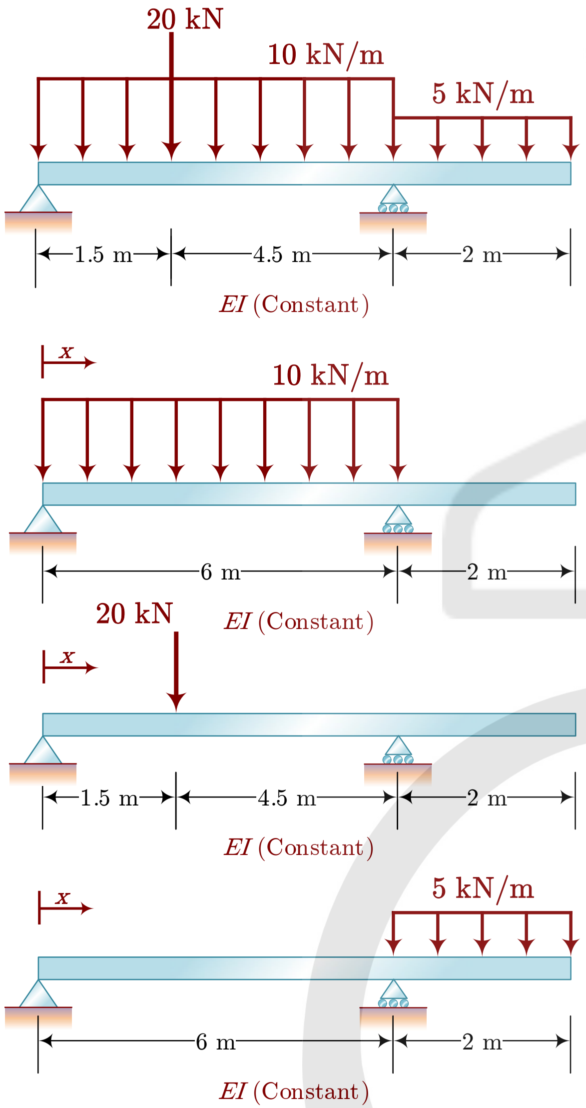
$$\Delta_{x2}=\frac{wx_2}{24EI}\left(l^3-2lx_2^2+x_2^3\right)=\frac{1.671\times10^{11}}{EI}$$

$$\Delta_{\max}=\max\left\{\left(\Delta_{\max 1}+\Delta_{x2}\right),\left(\Delta_{\max 2}+\Delta_{x1}\right)\right\}$$

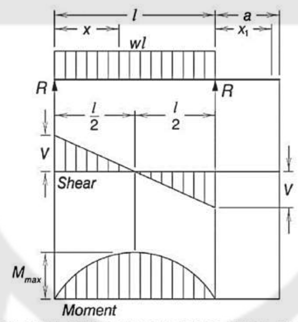
$$\max\left\{\left(\frac{7.742\times10^{10}}{EI}+\frac{1.671\times10^{11}}{EI}\right),\left(\frac{1.688\times10^{11}}{EI}+\frac{7.666\times10^{10}}{EI}\right)\right\}$$

$$=\max\left\{\left(\frac{2.445\times10^{10}}{EI}\right),\left(\frac{2.455\times10^{11}}{EI}\right)\right\}=\frac{2.455\times10^{11}}{EI}$$

Example (4): Determine the displacement at mid span between supports.



27. BEAM OVERHANGING ONE SUPPORT — UNIFORMLY DISTRIBUTED LOAD BETWEEN SUPPORTS

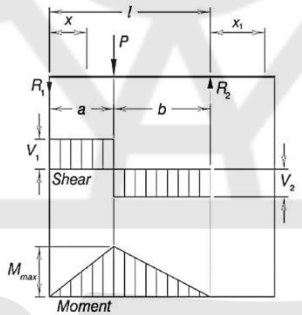


Δ_{max} (at center) .... $= \dfrac{5wl^4}{384EI}$

Δ_x .... $= \dfrac{wx}{24EI}(l^3 - 2lx^2 + x^3)$

Δ_{x_1} .... $= \dfrac{wl^3 x_1}{24EI}$

28. BEAM OVERHANGING ONE SUPPORT — CONCENTRATED LOAD AT ANY POINT BETWEEN SUPPORTS



Δ_{max} $\left(\text{at } x = \sqrt{\dfrac{a(a+2b)}{3}} \text{ when } a > b\right)$ .... $= \dfrac{Pab(a+2b)\sqrt{3a(a+2b)}}{27EIl}$

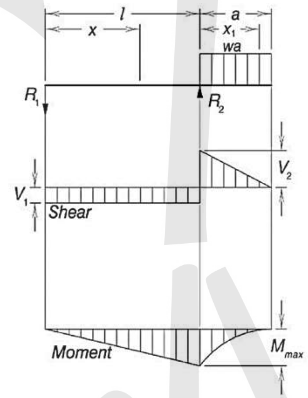
Δ_a (at point of load) .... $= \dfrac{Pa^2b^2}{3EIl}$

Δ_x (when $x < a$) .... $= \dfrac{Pbx}{6EIl}(l^2 - b^2 - x^2)$

Δ_x (when $x > a$) .... $= \dfrac{Pa(l-x)}{6EIl}(2lx - x^2 - a^2)$

Δ_{x_1} .... $= \dfrac{Pabx_1}{6EIl}(l + a)$

25. BEAM OVERHANGING ONE SUPPORT — UNIFORMLY DISTRIBUTED LOAD ON OVERHANG

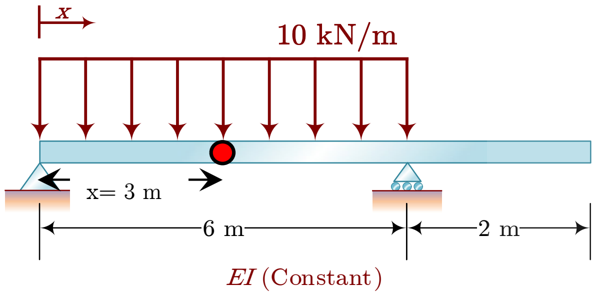


Δ_{max} $\left(\text{between supports at } x = \dfrac{l}{\sqrt{3}}\right) = \dfrac{wa^2l^2}{18\sqrt{3}EI} = 0.0321\dfrac{wa^2l^2}{EI}$

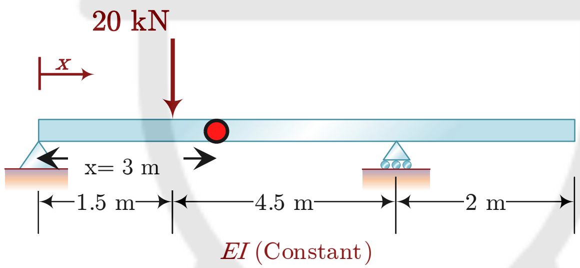
Δ_{max} (for overhang at $x_1 = a$)..... $= \dfrac{wa^3}{24EI}(4l + 3a)$

Δ_x (between supports) .... $= \dfrac{wa^2x}{12EIl}(l^2 - x^2)$

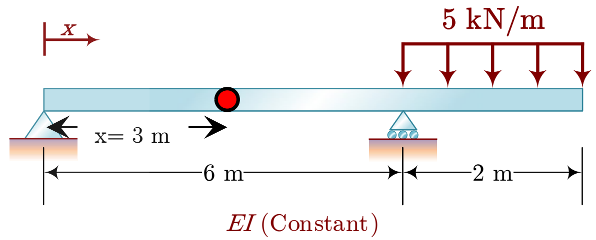
Δ_{x_1} (for overhang) .... $= \dfrac{wx_1}{24EI}(4a^2l + 6a^2x_1 - 4ax_1^2 + x_1^3)$



$$\Delta_{\max}\left(\text{at center}\right)=\frac{5wl^4}{384EI}$$



$$\Delta_x\left(\text{when } x>\text{a}\right)=\frac{Pa\left(l-x\right)}{6EIl}\left(2lx-x^2-a^2\right)$$



$$\Delta_x\left(\text{between supports}\right)=\frac{wa^2x}{12EIl}\left(l^2-x^2\right)$$

Displacement direction?

$$\Delta_{\max}\left(\text{at center}\right)=\frac{5wl^4}{384EI}$$ Down

$$\Delta_x\left(\text{when } x>\text{a}\right)=\frac{Pa\left(l-x\right)}{6EIl}\left(2lx-x^2-a^2\right)$$ Down

$$\Delta_x\left(\text{between supports}\right)=\frac{wa^2x}{12EIl}\left(l^2-x^2\right)$$ Up

$$\Delta_x\left(\text{between supports}\right)=\frac{5wl^4}{384EI}+\frac{Pa\left(l-x\right)}{6EIl}\left(2lx-x^2-a^2\right)-\frac{wa^2x}{12EIl}\left(l^2-x^2\right)$$

$$\Delta_x\left(\text{between supports}\right)=\frac{1.687\times10^{11}}{EI}+\frac{6.188\times10^{10}}{EI}-\frac{1.266\times10^{10}}{EI}=\frac{2.179\times10^{11}}{EI}$$

Questions?